Generalized Benney Lattice and the Heavenly Equation

A. Constandache and Ashok Das
Department of Physics and Astronomy,
University of Rochester,
Rochester, NY 14627-0171
USA

and

Ziemowit Popowicz
Institute of Theoretical Physics,
University of Wrocław,
50-205 Wroclaw
Poland.

Abstract

We generalize the Benney lattice and show that the new system of equations can be reduced to a generalized Chaplygin gas as well as the heavenly equation. We construct two infinite sets of conserved charges and show that one of the sets can be obtained from the Lax function. The conserved densities are related to Legendre polynomials and we present closed form expressions for the generating functions for these densities which also determines the Riemann invariants of the problem. We prove that the system is bi-Hamiltonian and that the conserved charges are in involution with respect to either of the Hamiltonian structures. We show that the associated generalized elastic medium equations are bi-Hamiltonian as well. We also bring out various other interesting features of our model.

1 Introduction:

The Benney lattice [1] is defined by a set of equations of the form

$$\frac{\partial A_k}{\partial t} = A_{k+1,x} + kA_{k-1}A_{0,x}, \qquad k = 0, 1, 2, \cdots \tag{1}$$

where the subscript x denotes derivative with respect to the x coordinate. Several physical systems can be obtained from the Benney lattice. In particular, note that if we identify [2]

$$A_k = u^k v, \qquad k = 0, 1, 2, \cdots \tag{2}$$

where u, v are two independent dynamical variables, then the set of equations in (1) reduces to

$$\frac{\partial v}{\partial t} = (uv)_x, \quad \frac{\partial u}{\partial t} = uu_x + v_x \tag{3}$$

which is the polytropic gas equation [3, 4, 5, 6, 7] with $\gamma = 2$. The Benney lattice defines a dispersionless system of equations [8] and can be given a Lax description in the following manner. Consider the Lax function

$$L = p + \sum_{k=0}^{\infty} A_k p^{-(k+1)} \tag{4}$$

where the coefficients, A_k, are functions of (x, t) while p represents the momentum of the classical phase space. Then, with the usual canonical Poisson brackets of classical mechanics, it is easy to verify that the Benney lattice is obtained from

$$\frac{\partial L}{\partial t} = -\frac{1}{2}\left\{(L^2)_+, L\right\} \tag{5}$$

where $()_+$ stands for the projection with non-negative powers of p. This shows that the system of equations in (1) is integrable. Furthermore, it is also known that this system of equations is Hamiltonian with three distinct Hamiltonian structures, much like the system of polytropic gas equations [4].

In this letter, we will generalize the Benney lattice and obtain a new system of equations which can also be identified with various physical systems. In particular, we will show that this system leads naturally to a generalization of the Chaplygin gas [4, 6] as well as the heavenly equation [9]. We will give a Lax description of this system and construct two infinite sets of conserved charges. One of the sets of charges follows directly from the Lax description while the connection of the second set to the Lax function remains unclear at the present. Interestingly, both sets of the charges are related to the Legendre polynomials. We also present the generating function for these charges as well as the Riemann invariants of the system. We construct two Hamiltonian structures associated with this system and show that the system is bi-Hamiltonian (is a pencil) [10]. This is quite easily seen in a suitable choice of the dynamical variables. We also show the involution of the charges directly from the structure of the generating functions. The generalized elastic medium equations, associated with this system, are also shown to be bi-Hamiltonian. We discuss the connection of the heavenly equation with this system of equations and conclude with some discussions on various other aspects of this sytem.

2 Generalized Benney Lattice

Let us consider a Lax function of the form

$$L = p^{-1} + \sum_{k=0}^{\infty} A_k p^{k+1} \tag{6}$$

If we assume the Poisson brackets of the classical phase space to be modified as [11]

$$\{A, B\}_m = p^m \{A, B\} \tag{7}$$

then, it is easy to check that the Lax equation

$$\frac{\partial L}{\partial t} = \frac{1}{2}\left\{(L^2)_{\leq 0}, L\right\}_m \tag{8}$$

does not lead to a consistent set of equations for $m = 0$ (namely, the standard Poisson bracket relations). For $m = 1$, we have a set of consistent equations provided all the odd coefficients, A_{2k+1}, in the Lax function vanish. Thus, taking our Lax function as

$$L = p^{-1} + \sum_{k=0}^{\infty} A_k p^{2k+1} \tag{9}$$

the Lax equation

$$\frac{\partial L}{\partial t} = \frac{1}{2}\left\{(L^2)_{\leq 0}, L\right\}_{m=1} \tag{10}$$

leads to a system of equations given by

$$\frac{\partial A_k}{\partial t} = A_{k+1,x} + (2k+1) A_k A_{0,x} \tag{11}$$

This system of equations can be thought of as a generalization of the Benney lattice.

Let us note next that this system of equations can be reduced to a generalized Chaplygin gas in the following manner. Let us identify

$$A_0 = u, \quad A_1 = -\frac{1}{2v^2}, \quad A_k = -\frac{1}{2}\sum_{i=0}^{k-1} A_{k-i-1} A_i, \quad k > 1 \tag{12}$$

where (u, v) are the independent dynamical variables. Then, it is straightforward to show that the system of generalized Benney lattice equations reduces to

$$\frac{\partial v}{\partial t} = -(uv)_x, \qquad \frac{\partial u}{\partial t} = u u_x + \frac{v_x}{v^3} \tag{13}$$

This looks very much like the Chaplygin gas (polytropic gas for $\gamma = -1$) [4, 6] except for the relative sign in the two equations. It can be seen easily that there is no transformation which will normalize the sign in both the equations.

The generalized Chaplygin equations have two infinite sets of conserved charges which can be constructed as follows. First, let us note that any conserved density, $h(u,v)$, of hydrodynamic type (namely, not depending on derivatives) must satisfy

$$\frac{dh}{dt} = \frac{\partial h}{\partial u}\, u_t + \frac{\partial h}{\partial v}\, v_t = \frac{dG}{dx} = \frac{\partial G}{\partial u}\, u_x + \frac{\partial G}{\partial v}\, v_x \tag{14}$$

From the forms of the equations as well as the integrability condition $G_{uv} = G_{vu}$, it follows that any conserved density of hydrodynamic type associated with the system must satisfy the relation

$$2u\frac{\partial^2 h}{\partial u \partial v} = v^{-3}\frac{\partial^2 h}{\partial u \partial u} + v\frac{\partial^2 h}{\partial v \partial v} \tag{15}$$

where the conserved charges are defined as

$$H = \int dx\, h \tag{16}$$

Let us also note from the structure of the Lax function that we can assign the canonical dimensions $[v] = 2, [u] = -2$ to the variables so that $[uv] = 0$. As a result, we can construct two independent sets of polynomials of the forms

$$h_k = u^k f_k(uv), \qquad \tilde{h}_k = v^k \tilde{f}_k(uv) \tag{17}$$

They must satisfy eq. (15) if they are to correspond to conserved densities of the system and it is not hard to check that this is indeed the case provided the functions $f(uv)$ and $\tilde{f}(uv)$ are related to the Legendre polynomials. Explicitly, f and $\tilde{f}$ have to satisfy the equations

$$\begin{aligned} z^2(1-z^2)f_k'' + 2z(k-(k+1)z^2)f_k' + k(k-1)f_k &= 0 \\ (1-z^2)\tilde{f}_k'' - 2z\tilde{f}_k' + k(k-1)\tilde{f}_k &= 0 \end{aligned} \tag{18}$$

where we have identified $z = uv$ and the primes denote derivatives with respect to z. The second equation is easily seen to correspond to the Legendre equation. With a change of variables, the first equation can also be seen to be related to the Legendre equation. The two infinite sets of conserved densities are obtained, in this way, to be

$$\begin{aligned} h_k &= 2^k u^k \frac{1}{(uv)^k}\, P_k(uv) = \left(\frac{v}{2}\right)^{-k}\, P_k(uv) \\ \tilde{h}_k &= \left(\frac{v}{2}\right)^{k+1}\, P_k(uv) \end{aligned} \tag{19}$$

where $k = 0, 1, 2, \cdots$ and $P_k(uv)$ represent the Legendre polynomials of order k. The first few densities in the two series have the explicit forms

$$\begin{aligned} h_0 &= 1 \\ h_1 &= 2u \\ h_2 &= 6\left(u^2 - \frac{1}{3v^2}\right) \end{aligned} \tag{20}$$

and

$$
\begin{aligned}
\tilde{h}_0 &= \frac{v}{2} \\
\tilde{h}_1 &= \frac{1}{4}uv^3 \\
\tilde{h}_2 &= \frac{1}{16}\left(3u^2v^5 - v^3\right)
\end{aligned} \tag{21}
$$

We will show next that the first set of conserved densities in (19) can be obtained from the Lax function. Let us note that, with the identification in eq. (12), we can sum the infinite series in (9) and the Lax function can also be written in the closed form

$$
L = p^{-1}\left(1 + 2up^2 + (u^2 - \frac{1}{v^2})p^4\right)^{\frac{1}{2}} \tag{22}
$$

There are several interesting things to note from this form of the Lax function. First of all, it is almost trivial to check from this form that we can write

$$
\text{Residue } (p^{-1}L^{2n}) = \left(\frac{v}{2}\right)^{-n} P_n(uv) = h_n \tag{23}
$$

Let us note that since the Lax function in (22) involves both positive and negative powers of p, as in the case of a non-standard Lax description of the polytropic gas [5, 6], it is natural to expect that the residues calculated around $p = 0$ and $p = \infty$ may provide the two sets of conserved densities. However, in this case, residues calculated at both these points yield the same conserved densities and we do not know yet how to relate the second set of conserved densities to the Lax function.

We also note here that the Lax function can also be expanded around $p = \infty$ so that we can write it in the form

$$
L = \sum_{k=-1} A_k p^{-(2k+1)} \tag{24}
$$

which has the same form as the Lax for the Benney lattice in (4). However, because of the difference in the classical Poisson brackets, the two systems are inequivalent. It is also interesting to point out here that the system of equations in (13) has a second Lax description. Consider, for example, the Lax function

$$
L = p\left(1 + 2up^{-2} + (u^2 - \frac{1}{v^2})p^{-4}\right)^{\frac{1}{2}} \tag{25}
$$

Then, it can be verified in a simple manner that the Lax equation

$$
\frac{\partial L}{\partial t} = \frac{1}{2}\left\{(L^2)_+, L\right\}_{m=1} \tag{26}
$$

also leads to the equations in (13). However, even this second Lax function yields only the first set of conserved densities.

Once we have the conserved densities in closed form and we know that they are related to the Legendre polynomials, we can write generating functions for these conserved densities in closed form. For example, it is easy to check that

$$T(u,v;\lambda) = \frac{v}{\sqrt{v^2 - 4uv^2\lambda + 4\lambda^2}} \tag{27}$$

where λ is an arbitrary constant parameter, generates the first set of conserved densities as

$$h_n = \left.\frac{\partial^n T(u,v;\lambda)}{\partial \lambda^n}\right|_{\lambda=0} \tag{28}$$

Similarly, the generating function for the second set of conserved densities has the closed form

$$\tilde{T}(u,v;\tilde{\lambda}) = \frac{v}{\sqrt{4 - 4uv^2\tilde{\lambda} + v^2\tilde{\lambda}^2}} \tag{29}$$

where $\tilde{\lambda}$ is an arbitrary constant parameter and we have

$$\tilde{h}_n = \left.\frac{\partial^n \tilde{T}(u,v;\tilde{\lambda})}{\partial \tilde{\lambda}^n}\right|_{\tilde{\lambda}=0} \tag{30}$$

As in the case of the polytropic gas, we find that the quantity inside the radical determines the Riemann invariants for the system [5, 6]. In this case, the two Riemann invariants have the forms

$$\lambda_{\pm} = \frac{v}{2}\left(uv \pm \sqrt{u^2v^2 - 1}\right) \tag{31}$$

The generalized Chaplygin gas is a bi-Hamiltonian system. We find that the system has the following two compatible Hamiltonian structures,

$$\begin{aligned}
\mathcal{D}_2 &= \frac{1}{8}\begin{pmatrix} 0 & \partial v(u^2v^2-1) \\ v(u^2v^2-1)\partial & -v(u^2v^2-1)\partial uv^3 - uv^3\partial v(u^2v^2-1)\end{pmatrix}\\
\mathcal{D}_3 &= \frac{1}{2}\begin{pmatrix} \partial(u^2-v^{-2}) + (u^2-v^{-2})\partial & 2(uv)_x \\ -2(uv)_x & \partial(v^2-u^2v^4) + (v^2-u^2v^4)\partial\end{pmatrix}
\end{aligned} \tag{32}$$

The equations (13) can be written in the Hamiltonian form with these structures as

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathcal{D}_3 \begin{pmatrix} \frac{\delta H_1}{\delta u} \\ \frac{\delta H_1}{\delta v} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta H_2}{\delta u} \\ \frac{\delta H_2}{\delta v} \end{pmatrix} \tag{33}$$

The compatibility of these two Hamiltonian structures can also be checked. However, it involves a lengthy calculation in these variables and takes a much simpler form in an alternate choice of variables. Therefore, we will discuss this question in the next section. However, we note here that, unlike the polytropic gas where there are three Hamiltonian structures, in this case, we have not been able to find the analog of the first Hamiltonian structure.

With these two Hamiltonian structures, the involution of the conserved charges can now be shown easily. In fact, it is much simpler to study the involution of the generating functions for the conserved densities themselves and show that, with respect to either of the structures,

$$
\begin{aligned}
\{T(u,v;\lambda),T(u,v;\lambda')\} &= K_x\\
\left\{\tilde{T}(u,v;\tilde{\lambda}),\tilde{T}(u,v;\tilde{\lambda}')\right\} &= L_x\\
\left\{T(u,v;\lambda),\tilde{T}(u,v;\tilde{\lambda})\right\} &= M_x
\end{aligned}
\tag{34}
$$

where K, L, M are complicated (but uninteresting) polynomials. This shows that the conserved charges (which are integrals of the densities), are in involution and, therefore, the system is completely integrable with respect to both the Hamiltonian structures.

We conclude this section by pointing out another interesting feature of these equations. We know that the polytropic gas and the elastic medium equations share the same Lax function, the same conserved charges and both these systems are bi-Hamiltonian with the same Hamiltonian structures [5]. In the present case, we also note that if we take the second series of conserved charges from (19) or (21), it is easy to verify that

$$
\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \frac{1}{8}\mathcal{D}_3 \begin{pmatrix} \frac{\delta \tilde{H}_1}{\delta u} \\ \frac{\delta \tilde{H}_1}{\delta v} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta \tilde{H}_0}{\delta u} \\ \frac{\delta \tilde{H}_0}{\delta v} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (u^2v^3 - v)_x \\ -(u^3v^6 - uv^4)_x \end{pmatrix}
\tag{35}
$$

It is clear, therefore, that the modified elastic medium equations, in this case, are also bi-Hamiltonian.

3 Heavenly Equation

Let us consider the Lax function in eq. (22) and identify

$$
a = (u^2 - \frac{1}{v^2}), \qquad b = 2u \tag{36}
$$

so that we can write

$$
L = p^{-1}(1 + bp^2 + ap^4)^{\frac{1}{2}} \tag{37}
$$

It is straightforward to check that the Lax equation,

$$
\frac{\partial L}{\partial t} = \frac{1}{2}\left\{(L^2)_{\leq 0}, L\right\}_{m=1} \tag{38}
$$

leads, in these variables, to the equations [11]

$$
a_t = ab_x, \qquad b_t = a_x \tag{39}
$$

These coupled set of first order equations can also be written as a second order equation of the form

$$\begin{aligned} (\ln a)_{tt} &= a_{xx} \\ \text{or,} \quad X_{tt} &= \left(e^{X}\right)_{xx} \end{aligned} \tag{40}$$

where we have defined $X = \ln a$. This is known as the heavenly equation (in $1+1$ dimensions) [9] and can be obtained as a continuum limit of the Toda lattice. This equation appears in the study of gravitational instantons [12] and can be linked to the Schröder equation which arises in the bootstrap models as well as in renormalization theory [13].

The description of the dynamical equations in terms of the variables, a, b, is simpler and brings out many features rather nicely. For example, the two Hamiltonian structures (32) can be written in these variables to be

$$\begin{aligned} \mathcal{D}_2 &= \frac{1}{2}\begin{pmatrix} 0 & a\partial \\ \partial a & 0 \end{pmatrix} \\ \mathcal{D}_3 &= 2\begin{pmatrix} \partial a^2 + a^2\partial & a\partial b \\ b\partial a & \partial a + a\partial \end{pmatrix} \end{aligned} \tag{41}$$

We note that the first two nontrivial Hamiltonian densities of eq. (20) take the forms, in these variables, as

$$h_1 = \frac{b}{2}, \qquad h_2 = b^2 + 2a \tag{42}$$

and it is straightforward to check that the above equations can be written in the Hamiltonian forms

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \mathcal{D}_3 \begin{pmatrix} \frac{\delta H_1}{\delta a} \\ \frac{\delta H_1}{\delta b} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta H_2}{\delta a} \\ \frac{\delta H_2}{\delta b} \end{pmatrix} \tag{43}$$

It is clear that these Hamiltonian structures have a much simpler form than the ones given in the last section, although they are equivalent to these. More importantly, we note now that under a shift $a \to a, b \to b + \lambda$, where λ is an arbitrary constant parameter

$$\mathcal{D}_3 \longrightarrow \mathcal{D}_3 + 4\lambda\mathcal{D}_2 \tag{44}$$

This shows that the two structures are compatible and that the system is bi-Hamiltonian (otherwise also known as a pencil system). The integrability of this system, therefore, follows. This also brings out why it is more involved to explicitly see the compatibility of the two structures in the old variables, namely, the appropriate shift is highly nontrivial in the (u, v) variables.

Furthermore, since the Hamiltonian structures have a simpler form in these variables, the recursion operator can also be constructed easily. It has the form

$$\mathcal{R} = \mathcal{D}_2^{-1}\mathcal{D}_3 = 4\begin{pmatrix} b - a^{-1}\partial^{-1}ab_x & 2 - a^{-1}\partial^{-1}a_x \\ 2a & b \end{pmatrix} \tag{45}$$

The recursion operator, in the old variables can, of course, be obtained from this through a coordinate redefinition. However, the form is much more complicated and is uninteresting to list here.

4 Summary and Discussions

In this letter, we have generalized the Benney lattice which can describe a generalized Chaplygin gas equation. We have constructed the Lax representation for this system and have constructed two infinite sets of conserved charges which are related to the Legendre polynomials. We can relate one of these two sets to the residues of the Lax function and the relation of the second set to the Lax is unclear at the present. We have given closed form expressions for the generating functions for these densities which also leads to the two Riemann invariants for the system. We have shown that the system is bi-Hamiltonian (a pencil system) and have shown the involution of charges thereby proving complete integrability of the system. We have also shown that the associated generalized elastic medium equations are bi-Hamiltonian as well. This system of generalized Chaplygin gas equations can be transformed to the heavenly equation which arises in many branches of physics. Many features of the system take a simpler form in this description.

We would now like to make some general comments on some other aspects of our system. First, we note that the higher order equations of the hierarchy of generalized Chaplygin gas can be obtained from

$$\frac{\partial L}{\partial t_n} = \frac{1}{2n}\left\{(L^{2n})_{\leq 0}, L\right\}_{m=1}, \qquad n = 1, 2, \cdots \tag{46}$$

where L is the Lax function in (9) or (22). However, in addition, we have also found that the system of equations

$$\frac{\partial L}{\partial t} = \frac{1}{2n+1}\left\{(L^{2n+1})_{\leq 0}, L\right\}_{m=2} \tag{47}$$

also leads to consistent equations. However, we have not analyzed integrability properties of such systems completely. It is also worth noting that the Lax function in (6) leads through

$$\frac{\partial L}{\partial t} = \frac{1}{2}\left\{(L^{2})_{\leq 0}, L\right\}_{m=2} \tag{48}$$

to the Benney lattice. Finally, without going into details, we would like to note that we can generalize the Lax function in (22) to

$$L = p^{1-n}\left(1 + A_1 p^n + A_2 p^{2n}\right)^{\frac{1}{n}} \tag{49}$$

This will lead to a generalized set of equations

$$\begin{aligned} A_{1,t} &= \frac{2(n-1)}{n}\left(A_{2,x} - \frac{(n-2)}{n}A_1 A_{1,x}\right) \\ A_{2,t} &= 2\left(-\frac{(n-3)}{n}A_2 A_{1,x} + \frac{(n-2)}{n^2}A_1 A_{2,x}\right) \end{aligned} \tag{50}$$

following from the Lax equation

$$\frac{\partial L}{\partial t} = \frac{1}{n}\left\{(L^{2})_{\leq 0}, L\right\}_{m=n-1} \tag{51}$$

All these equations are likely to be integrable since they follow from a Lax description. However, we have not studied these systems in more detail.

Acknowledgments

This work was supported in part by US DOE grant no. DE-FG-02-91ER40685 as well as by NSF-INT-0089589.

References

[1] D. J. Benney, Stud. Appl. Math. **52** (1973) 45.

[2] Y. Nutku and M. Pavlov, J. Math. Phys. **43** (2002) 1441.

[3] G. B. Whitham, *Linear and Nonlinear Waves*, (Wiley, New York, 1974).

[4] P. J. Olver, Y. Nutku, J. Math. Phys. 29 (1988) 1610; M. Arik, F. Neyzi, Y. Nutku, P. J. Olver and J. Verosky, J. Math. Phys. **30** (1988) 1338.

[5] J. C. Brunelli, A. Das, Phys. Lett. A235 (1997) 597.

[6] J. C. Brunelli, A. Das, Phys. Lett. B426 (1998) 57.

[7] A. Constandache, A. Das, F. Toppan, *Lucas polynomials and a standard Lax representation for polytropic gas dynamics*, hep-th/0110097, to be published in Lett. Math. Phys.

[8] B. A. Kupershmidt, Yu. I. Manin, Funct. Anal. Appl. **11** (1978) 188, *ibid* **12** (1978) 20; J. Gibbons and Y. Kodama in, *Singular limits of dispersive waves*, ed. N. Ercolani et al, Plenum (1994); J. Gibbons, Phys. Lett. **A90** (1982) 7; D. B. Fairlie, I. A. Strachan, Inverse Prob. **12** (1998) 885.

[9] J. F. Plebanski and H. Garcia-Compean, Acta Phys. Polon. **B26** (1995) 3, Int. J. Mod. Phys. **A10** (1995) 3371; E. Alifanto, G. Soliani and L. Solombrino, Lett. Math. Phys. **41** (1997) 379.

[10] F. Magri, J. Math. Phys. **19** (1978) 1156; A. Das, *Integrable models*, World Scientific (1989); M. Błaszak, *Multi-Hamiltonian theory of dynamical systems*, Springer-Verlag (1998).

[11] M. Gürses and K. Zheltukhin, J. Math. Phys. **42** (2001) 1309.

[12] T. Eguchi, P. B. Gilkey and A. J. Hanson, Phys. Reps. **66** (1980) 213; S. W. Hawking in, *Recent developments in gravity*, ed. M. Levy and S. Deser, Plenum (1979).

[13] R. Hagedorn and J. Rafelski, Comm. Math. Phys. **83** (1982) 563.